Fano fourfolds with large anticanonical base locus

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UNIVERSITÀ DEGLI STUDI DI MILANO

Introduction

Joint work with Andreas Höring

X Fano manifold $\rightsquigarrow$ X smooth proj. variety (over $\mathbb{C}$) s.t. $-K_X$ is ample

Classification completed up to dimension 3:

- **dim = 1 :** $\mathbb{P}^1$
- **dim = 2 :** del Pezzo surfaces, 10 families
- **dim = 3 :** 105 families
- **dim ≥ 4 :** Partial results
- Buondedness $\rightsquigarrow$ finite number of families for each fixed dimension

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  - · Mori-Mukai, $\rho_X \geq 2$
  - · Iskovskikh, $\rho_X = 1$
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$-K_X$ is ample $\rightsquigarrow$ $-mK_X$ is very ample for $m \gg 0$, so

$$\begin{cases} \dim H^0(X, -mK_X) > 0 \\ -mK_X \text{ is globally generated, i.e. } \operatorname{Bs}|-mK_X| = \emptyset \end{cases}$$

Question: does the same hold for $-K_X$?

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Q1: is $H^0(X, -K_X) \neq 0$?

Ambro-Kawamata conjecture (smooth case)

Let X be a smooth proj. var. and let A an ample Cartier divisor. If $A - K_X$ is ample, then $H^0(X, A) \neq 0$.

X is smooth Fano and $\dim(X) \leq 4 \implies \dim H^0(X, -K_X) \geq 2$.
(Riemann-Roch computation)

Q2: is $-K_X$ always globally generated?

No! There are examples where $\mathrm{Bs}\,|-K_X| \neq \emptyset$.

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<u>No</u>! There are examples where $\mathrm{Bs}\,|-K_X| \neq \emptyset$.

Examples (dimension 2)

Ex1

Let S_1 be a smooth del Pezzo surface of degree 1, i.e. $S_1 := \mathrm{Bl}_8 \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$.

• $\mathrm{Bs}|-K_{S_1}| \neq \emptyset$: $|-K_{S_1}|$ consists of the transforms of plane cubics C passing through the blown-up points $\rightsquigarrow$

$\exists$ a point $P \in S_1$ s.t. $\pi(P) \in C$ for all $C \implies \mathrm{Bs}|-K_{S_1}| = \{P\}$.

• a general element of $|-K_{S_1}|$ is smooth.

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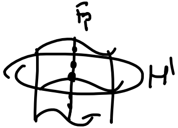
Examples (dimension 3)

Ex2 Z Fano, $-K_Z \sim 2H$, H ample, $H^3 = 1$, $Z \subseteq \mathbb{P}(1,1,1,2,3)$ sextic

Let Z be a del Pezzo 3-fold of degree 1, and let $X := \mathrm{Bl}_\Gamma Z \xrightarrow{\pi} Z$ the blow-up along an elliptic curve $\Gamma \implies X$ is a smooth Fano 3-fold and $\mathrm{Bs}\,|-K_X|$ is a non-trivial π-fibre. ($\cong \mathbb{P}^1$)

$-K_Z$ is s.g., $\mathrm{Bs}\,|-\frac{1}{2}K_Z| = \{p\}$. $H \in |-\frac{1}{2}K_Z|$ general, $H \sim S_1$



$\mathrm{Bs}\,|-K_X| \cap H' = \mathrm{Bs}\,|-\frac{1}{2}K_Z| \cap H = p$

$H \cong H'$

$F_p \subseteq \mathrm{Bs}\,|-K_X|$ $(\overset{\uparrow}{=})$

Examples (dimension 3)

Ex3

$\mathrm{Bs}|-K_{S_1}|$

Let $X = S_1 \times \mathbb{P}^1 \implies X$ is a smooth Fano 3-fold, $\mathrm{Bs}\,|-K_X| = \{P\} \times \mathbb{P}^1$.

Fact:

$X = Y \times T$. $\quad -K_X = \pi_Y^*(-K_Y) + \pi_T^*(-K_T)$

$\mathrm{Bs}|-K_X| = \mathrm{Bs}|-K_Y| \times T \cup Y \times \mathrm{Bs}|-K_T|$

- [Shokurov '80] If X is a smooth Fano 3-fold s.t. $\mathrm{Bs}\,|-K_X| \neq \emptyset$, then $\mathrm{Bs}\,|-K_X| \cong \mathbb{P}^1$ and a general element of $|-K_X|$ is smooth.
- **Ex2**, **Ex3** are the only examples and $\mathrm{codim}(\mathrm{Bs}\,|-K_X|) = 2$.

Some examples in dimension 4

Ex4 [Casagrande, Druel '15]

- Z del Pezzo 3-fold of degree 1, $(a, d) = (0, 1), (1, 2)$
- $D \in |\mathcal{O}_Z(d)|$ smooth
- $Y = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$, G a section with $\mathcal{N}_{G/Y} \cong \mathcal{O}_Z(a)$
- $\sigma \colon X \to Y$ blow-up along $G \cap \pi^{-1}(D) \rightsquigarrow E = \mathrm{Exc}(\sigma)$

$$X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z$$

Then, X is a smooth Fano 4-fold.

[S. '23]:

- $\dim(\mathrm{Bs}|-K_X|) = 0$ and $\mathrm{Bs}|-K_X|$ is either 1 or 2 points
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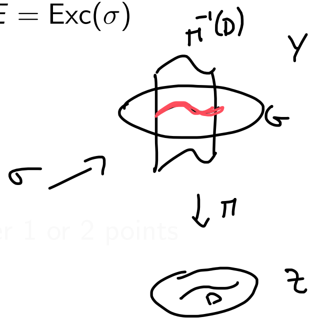

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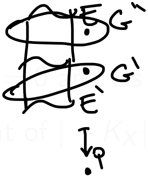

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$\{p\} = \mathrm{Bs}\,|-\frac{1}{2}K_Z|$

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Ex5 [Casagrande, Codogni, Fanelli '19]

Let $X' := \mathrm{Bl}_8\,\mathbb{P}^4$. Its Fano model X is the moduli space $M_{S_1,-K_{S_1}}$ of semistale torsion-free sheaves of rank 2, $c_1 = -K_{S_1}$ and $c_2 = 2$:

$$\exists \text{ a SQM } X \dashrightarrow X' \text{ s.t. } X \text{ is a smooth Fano 4-fold.}$$

- [Xie '22] $\boldsymbol{Bs}\,|-K_X| \cong \mathbb{P}^1$ and a general element of $|-K_X|$ is smooth.

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sequence of flips

- [Xie '22] $\boldsymbol{Bs}\,|-K_X| \cong \mathbb{P}^1$ and a general element of $|-K_X|$ is smooth.

General results

Strong Bertini's theorem [Diaz, Harbater '91]

Let X be a smooth proj. var. and let $\mathfrak{d}$ be a linear system. If $\boldsymbol{Bs}\,|\mathfrak{d}|$ is smooth and $\dim(\boldsymbol{Bs}\,|\mathfrak{d}|) < \frac{1}{2}\dim(X)$, then a general element in $\mathfrak{d}$ is smooth.

X Fano 4-fold. If $\underline{Bs}\,|-K_X|$ is smooth of dim ≤ 1

$\Rightarrow$ $Y \in |-K_X|$ general is smooth

General results

[Kawamata '00; Höring, Voisin '11; Heuberger '15]

Let X be a smooth Fano 4-fold. Then:

- $h^0(X, -K_X) \geq 2$;
- a general $Y \in |-K_X|$ is a normal prime divisor with at most terminal Gorenstein singularities.

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- a general $Y \in |-K_X|$ is a normal prime divisor with at most terminal Gorenstein singularities.

- A general $Y \in |-K_X|$ is irreducible + $h^0(-K_X) \geqslant 2$ $\Rightarrow$ $\dim Bs|-K_X| \leq 2$
- Y is $\mathbb{Q}$-factorial iff Y is factorial
- When look at known examples, it's not known if $\exists$ X Fano 4-fold s.t. $h^0(-K_X) = 2$.

General results

What about Fano manifolds X s.t. $\dim(\mathrm{Bs}|-K_X|) \geq \frac{1}{2}\dim(X)$?

Lemma 1

Let X be a Fano manifold, and let $B \subseteq \mathrm{Bs}|-K_X|$ of dimension at least $\frac{1}{2}\dim(X)$. If the conormal bundle $\mathcal{N}^*_{B/X}$ is nef, then every $Y \in |-K_X|$ has at least one singular point along B.

- $H^0(X, -K_X) \simeq H^0(X, \mathcal{I}_B \otimes -K_X) \xrightarrow{\alpha} H^0(B, \mathcal{I}_B/\mathcal{I}_B^2 \otimes -K_X)$, where $\mathcal{I}_B/\mathcal{I}_B^2 = \mathcal{N}^*_{B/X}$

$s \in H^0(X,-K_X) \leftrightarrow Y$

Y is smooth along B iff $\alpha(s)$ is nowhere vanishing iff $c_d(\mathcal{N}^*_{B/X} \otimes -K_X) = 0$

$d = \mathrm{codim}_X B \leq \dim B$ and $\mathcal{N}^*_{B/X} \otimes -K_X$ ample $\Rightarrow$ ⨳

Ex6 $\xrightarrow{\pi} S_1$

Let $X := S_1 \times S$, S any smooth del Pezzo surface.

- $B = \{P\} \times S \subseteq \mathrm{Bs}\,|-K_X|$ and $\mathcal{N}^*_{B/X} \cong \mathcal{O}_B^{\oplus 2} \rightsquigarrow$ Lemma 1 applies.

$B = \pi^{-1}(P)$

$S_1 \times S_1$ $\quad$ $\mathrm{Bs}\,|-K_X| = \{P\} \times S \cup S \times \{P\}$

Y is sing along $\{P\} \times S$

Ex7

Let Z' be **Ex2**: $Z' \xrightarrow{\pi} Z$ is the blow-up of a del Pezzo 3-fold Z of degree 1 along an elliptic curve Γ. Let $X := Z' \times \mathbb{P}^1$, $C := \mathrm{Bs}\,|-K_{Z'}| \cong \mathbb{P}^1$ and $B := \mathrm{Bs}\,|-K_X|$.

- $B = C \times \mathbb{P}^1$, $\mathcal{N}^*_{C/Z'} \cong \mathcal{O}_C \oplus \mathcal{O}_C(1) \rightsquigarrow \mathcal{N}^*_{B/X}$ is nef and Lemma 1 applies.

More examples in dimension 4

Ex8 (Generalization of **Ex2**)

For $n \geq 4$, let M be a general sextic hyp. in $\mathbb{P}(1^n, 2, 3) \rightsquigarrow M$ is a del Pezzo n-fold of degree 1, i.e. $-K_X \sim (n-1)A$ with A ample.

- $|A|$ has a unique base point P; let Γ be the complete intersection of $n-1$ general element in $|A| \rightsquigarrow \Gamma$ is a smooth elliptic curve.

- Let $X := \mathrm{Bl}_\Gamma M \to M$ be the blow-up along $\Gamma \implies X$ is smooth Fano and $\mathrm{Bs}\,|-K_X|$ contains a surface $B \cong \mathbb{P}^{n-2}$ s.t. $\mathcal{N}^*_{B/X} \cong \mathcal{O}_B \oplus \mathcal{O}_B(1) \rightsquigarrow$ Lemma 1 applies.

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$\Gamma = D_1 \cap \dots \cap D_{n-1}$, $D_i \in |A|$. $V = |\langle D_1, \dots, D_{n-1} \rangle|$, $Bs(V) = \Gamma$

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$X \xrightarrow{\varphi} \mathbb{P}^{n-2}$ $\quad -K_X \quad$ $B = \pi^{-1}(P)$

$\pi \downarrow$

$M \ni P$

Motivation

- **Ex6**, **Ex7** and **Ex8** are the only known examples of Fano 4-folds s.t. $\dim \mathrm{Bs}\,|-K_X| = 2$;
- in these examples, a general $Y \in |-K_X|$ is not $\mathbb{Q}$-factorial!

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$B \subseteq \mathrm{Bs}|-K_X|$ (surf)

B is smooth, while Y is sing along B

If Y is $\mathbb{Q}$-factorial (thus fact.), B would be Cartier

Main result

Theorem [Höring, S. '23]

Let X be a smooth Fano 4-fold such that $h^0(X, -K_X) \geq 3$. Assume that $\mathrm{Bs}\,|-K_X|$ is an irreducible normal surface B. Then, a general element $Y \in |-K_X|$ is not $\mathbb{Q}$-factorial, in particular it is singular.

Remarks

- The theorem covers all know examples.
- Does this hold assuming that $\mathrm{Bs}\,|-K_X|$ *contains* a surface?
- Are there smooth Fano 4-folds s.t. $h^0(X, -K_X) = 2$?

In general, B is a semi-log canonical (slc) surface:

- Can we remove the hypothesis of normality?

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- Can we remove the hypothesis of normality?

Sketch of proof

Setup.

Let $Y \in |-K_X|$ general $\rightsquigarrow$ Y is an irreducible normal CY 3-fold. The restriction map

$$H^0(X, -K_X) \twoheadrightarrow H^0(Y, -K_X) \rightarrow H^1(\mathcal{O}_X) = 0$$

is surjective. Thus, $\mathrm{Bs}\,|-K_{X|Y}| = \mathrm{Bs}\,|-K_X| = B$.

X Fano,

Moreover, $h^0(Y, -K_X) = h^0(X, -K_X) - 1 \geq 2 \rightsquigarrow$ if $|M|$ is the mobile part of $|-K_{X|Y}|$, then

$$-K_{X|Y} \sim M + B.$$

- We argue by contradiction and assume that Y is $\mathbb{Q}$-factorial.

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Sketch of proof

dim $X = 4$

- Y is factorial $\rightsquigarrow$ M and B are Cartier divisor of Y.
- By Lefschetz hyperplane theorem, $\exists!\, M_X,\ B_X \in \mathrm{Pic}(X)$ such that

$$M \sim M_{X|Y}, \quad B \sim B_{X|Y}.$$

? Are M_X, B_X effective?

- [Kollár '91]: M_X (resp. B_X) is nef $\iff$ M (resp. B) is nef.

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This is equivalent to

$$\overline{NE}(Y) \hookrightarrow \overline{NE}(X)$$

$\cong$

Sketch of proof: not nef case

Assume M_X not nef: use the embedding $Y \subset X$ and look at M_X-negative elementary contractions.

- B_X is not nef; if B_X effective, then $h^0(X, B_X) = 1$
- $-K_X + M_X$ is nef & big $\rightsquigarrow$ control cohom. of M_X
- M_X is effective and mobile
- $\exists$ a smooth rational curve ℓ s.t. $M \cdot \ell < 0 \implies \ell \subset B$
- technical result on irregular surfaces $\rightsquigarrow$ $M_{|B}$ does not contain rational curves. **Contradiction!**

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  $q(B) \geq 1$
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Sketch of proof: nef case

Assume M_X nef: then M_X is semiample and $\exists$ a morphism with connected fibres

$$\varphi_{|mM_X|} = \varphi : X \to T$$

s.t. T is normal and $M_X \sim \varphi^* M_T$ for an ample Cartier divisors M_T on T.

- case by case study based on dim T:
  - dim $T > 1$
  - dim $T < 3$
  - dim $T \neq 2$ (most difficult part)
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Thank you!